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Weighted Shapley values of efficient portfolios

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Abstract. Shapley value theory, which originally emerged from cooperative game theory, was established for the purpose of measuring the exact contribution of agents playing the game. Subsequently, the Shapley value was used in finance to decompose the risk of optimal portfolios, attributing to the various assets their exact contribution to total risk and return. In the present paper, the Shapley value results of Shalit [*Annals of Finance* **17**(1) (2021), 1–25] are extended by using weighted Shapley values to decompose the risk of optimal portfolios. The weighted concept, as axiomatized by Kalai and Samet [*Journal of Game Theory* **16**(3) (1987), 205–222], provides a solution to cooperative games when the symmetry of players cannot be justified. The weighted Shapley value theory is applied to model efficient mean-variance portfolios and price their constituents. The computation is carried out for the 13 most traded US stocks in 2020 and the results are compared with the standard Shapley values.

Keywords: Mean- variance portfolios, trading volume, risk valuation

1. Introduction

Several years ago, the Shapley value [21] was applied to decompose the risk of optimal portfolios, attributing to the assets their exact contribution to portfolio risk and return. The present paper ensues from the concept of using the Shapley value in financial theory and risk allocation which is quite prevalent in cost sharing, optimal profit distribution, and risk attribution as evidenced by the results of [10,12,24,26] to cite only a few. Using the Shapley value in portfolio theory, however, has been more limited. Only recently, was it applied to portfolio risk allocation, particularly to efficient portfolios that weight risk vs return as developed by Ortmann [13] and Colin-Baldeschi et al. [5] who used Shapley theory to price the market risk of individual assets. More recently, Simonian [23] used the Shapley value to construct optimal portfolios.

Shapley value theory, which emerges from cooperative game theory, is applied for the purpose of measuring the exact attribution to agents playing the game. In a cooperative game, players interact in order to optimize a common objective whose utility is transferable. One of the less used properties of the Shapley value as assessed by Roth [15] is that the value represents a von Neumann-Morgenstern utility for a risk-neutral individual[1]. This result implies that the only value accepted by risk-averse and risk-lover investors is the Shapley value that prices correctly securities in a financial market. The notion of implementing the Shapley value to decompose inequality measures by sources of income was formulated by Shorrocks [22], although the paper was first circulated in 1999. The same approach was further developed by Sastre and Trannoy [18]. This income inequality decomposition theory was applied both to financial risk and portfolios, being that inequality and risk measures are known to be closely related. This task was performed by Terraza and Mussard [25] and [11] who were the first to extract the Shapley value of simple portfolios. They followed Shorrocks [22] formulation to decompose the covariance between two securities to assess the contribution of each security to portfolio risk.

In the present paper, the Shapley value results formulated by Shalit [20] are extended by using weighted Shapley values to decompose the risk of optimal portfolios. The concept, as devised by Shapley [21] and Owen [14] and axiomatized by Kalai and Samet [9], provides

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[1] This observation was pointed out to me by an anonymous referee.

a solution to cooperative games when players symmetry cannot be justified. This is evident with portfolios of company stocks that are rarely interchangeable. Indeed, financial markets are characterized by shares that are traded considerably more than others. This observation leads to contemplate that trading volume differential impacts portfolio risk. First, I present the weighted Shapley value theory and apply it to optimal mean-variance portfolios. Then, I compute the weighted Shapley values for the 13 most traded US stocks in 2020 when assets weights used in the model are approximated by the stocks trading volumes. I compare the new results with the standard Shapley values.

2. On weighted Shapley values

Our portfolio is viewed as a cooperative game played by its assets in order to minimize risk for specific mean returns. The Shapley value is measured *in terms of units of risk* to extract the exact contribution of each stock to the optimal portfolio. Shapley value theory ensures that the risk attributed to the various assets in the portfolio is *anonymous*, so that the marginal contributions are independent of the order in which assets are added to or removed from the portfolio and *exact* in the sense that all participants bear the entire risk. Expressed as a solution to cooperative games, the Shapley value has been commonly characterized by a series of axioms, namely: efficiency, additivity, dummy player, and symmetry[2]. The efficiency axiom requires that the value of the game is the sum attributed to the players. The dummy player axiom implies that a null player does not add value to the game. The additivity axiom demands that adding the values of two games is the value of the combined games. Symmetry requires that different game participants are treated identically if their individual value is the same.

The last axiom is the most problematic as players tend to be heterogeneous and would like to use their idiosyncrasies to extract some additional benefits. This seems to be the case in portfolio analysis where shares are quite divisible but corporations cannot always be structurally compared. Before presenting the weighted Shapley value concept I discuss the more familiar Shapley value model to evaluate an investment model.

Consider a stock market game whose purpose is to minimize portfolio risk expressed by the variance. For a set N of n securities, the Shapley value is calculated from the contribution of each and every security in the portfolio. To capture the symmetric and exact way each security contributes to the complete portfolio, we compute the risk v for each and every subset of stocks $S \subset N$. In total, we have 2^N portfolios or coalitions including the empty set.

We next compute the marginal contribution of each security to the risk of the subset portfolio. For a given portfolio S, security k contributes marginally to the subset by $v(S) - v(S \setminus \{k\})$, where $v\,(S)$ is the risk of portfolio S, and $v(S \setminus \{k\})$ is the risk of the portfolio S without security k. Portfolios are predefined and all the orderings are equally probable. Hence, $S \setminus \{k\}$ is the portfolio that precedes k, and its contribution to coalition S is computed when all the orderings of S are accounted for. Given equally probable orderings, we compute their expected marginal contribution. Therefore, we need the probability that, for a given ordering, the subset $S \subset N$, $k \subset S$ is seen as the union of security k and all the securities that precede it. Two probabilities are used here: First, the probability that k is in s (s being the number of stocks in S) being equal to $1/n$, and second, that $S \setminus \{k\}$ arises when $s-1$ securities are randomly chosen from $N \setminus \{k\}$, that is $(n-s)!\,(s-1)!\,/(n-1)!$.

The Shapley value for security k is obtained by averaging the marginal contributions to the risk of all portfolios for the set of N securities and the risk function v, which in mathematical terms is written as

$$Sh_k(N, v) = \sum_{S \subset N, k \subset S} \frac{(n-s)!\,(s-1)!}{n!} [v(S) - v(S \setminus \{k\})] \tag{1}$$

or, alternatively,

$$Sh_k(N, v) = \sum_{S \subset N, k \subset S} \frac{s!\,(n-s-1)!}{n!} [v(S \cup \{k\}) - v(S)]. \tag{2}$$

Summing up the Shapley values of all the assets in the portfolio equals its total risk, namely,

$$v(N) = \sum_{k=1}^{N} Sh_k(N, v). \tag{3}$$

These equations are the basic formulas needed to calculate the Shapley values.

Shapley [21] himself was aware of the lack of symmetry that existed between players and therefore

[2] For a theoretical presentation of the Shapley value concept see [16].

proposed the concept of weighted values by providing exogenously given weights. For [9,14,21] all of whom developed the weighted value these factors were understood as bargaining power of the players. On this basis, for portfolio analysis, I suggest using the trading volume of the assets in the portfolio as weights. The justification for this choice is the interaction between trading volume and systematic risk as evidenced by Ciner [4] and Hrdlicka [7]. Indeed assets with larger trading volume can be seen as more powerful since high trading volume indicates higher liquidity and as a facility for short and long trading.

I now present the concept of weighted Shapley values as developed by Kalai and Samet [9]. There is a considerable literature on the axiomatization of weighted Shapley values owing to the asymmetries that exist between the players. I have chosen to interpret these weights as precondition for bargaining power or some inherited valuation due to age, function, history, etc. For this purpose, let $\lambda = \{\lambda_i, i \in N\}$ be a set of non-negative weights associated with the players. The immediate insight is to use these weights and construct probabilities in order to compute the weighted Shapley values that yield the relative weights:

$$\varphi_i = \frac{\lambda_i}{\sum_j^n \lambda_j}. \tag{4}$$

We can now express the weighted Shapley values by using the relative weights of Eq. (4) with the Shapley value in Eq. (2) to yield the probabilistic formula:

$$Sh_k(N, v) = \sum_{S \subset N, k \in S} \frac{\lambda_k}{\sum_j^n \lambda_j} \frac{s!\,(n-s-1)!}{n!} [v(S \cup \{k\}) - v(S)]. \tag{5}$$

Recently, I was made aware by a referee of the close relationship between the weighted Shapley value and the family of least-square values of Ruiz et al. [17]. The latter value is obtained by calculating the weighted variance of all the coalitions marginal contributions participants which is basically yields the weighted Shapley value concept as a special case. Still, I have preferred to follow the Shapley value in my application to portfolio analysis.

3. The Shapley value of efficient portfolios

I now present the Shapley value of portfolio assets on the mean-variance efficient frontier that is the set of portfolios that minimize risk for a given mean. Since Shapley value theory works best with a single attribute imputed to all game participants, I use optimal portfolios whose expected returns are always at their minimum risk. Before presenting the weighted Shapley value concept I discuss the more familiar Shapley value model to evaluate an efficient investment as developed by Shalit [20].

Let us consider the set of frontier portfolios generated by minimizing the portfolio variance for a given expected return. To construct a portfolio frontier, consider N risky assets with returns $\mathbf{r}$ that are linearly independent implying that the variance-covariance matrix of asset returns Σ is non-singular. Denote by μ the vector of the asset's expected returns, and by $\mathbf{w}$ the vector of portfolio weights such that $\sum_{i=1}^N w_i = 1$. Assume $\mathbf{w} \gtreqless 0$, thereby allowing for short sales. An efficient portfolio is obtained by minimizing the variance portfolio σ_p^2 subject to a required mean μ_p. We minimize $\frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ subject to $\mu_p = \mathbf{w}'\mu$ and the portfolio constraint $1 = \mathbf{w}'\mathbf{l}$, where $\mathbf{l}$ is an N-vector of ones. Following [8], the solution is obtained by minimizing the Lagrangian with the two constraints and deriving the first-order conditions (FOC) for a minimum, as the second-order conditions are satisfied by the non-singularity of Σ.

Define the quadratic forms: $A = \mathbf{l}'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\mu$, $C = \mathbf{l}'\Sigma^{-1}\mathbf{l}$, and $D = BC - A^2$. The scalars B and C are positive since matrix Σ is positive-definite and so is its inverse. Scalar D is also positive following Huang and Litzenberger's [8] claim. From the FOC for a minimum variance, the optimal portfolio weights for a given mean μ_p are derived as

$$\mathbf{w}_p^* = \frac{1}{D}[B \cdot \boldsymbol{\Sigma}^{-1}\mathbf{l} - A \cdot \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}] + \frac{1}{D} [C \cdot \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - A \cdot \boldsymbol{\Sigma}^{-1}\mathbf{l}]\mu_p. \tag{6}$$

As the frontier portfolios are delineated in the standard deviation-mean space, their variance for a given μ_p is formulated by

$$\sigma_p^2 = \mathbf{w}_p'\boldsymbol{\Sigma}\mathbf{w}_p = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)^2 + \frac{1}{C}. \tag{7}$$

Equation (7) represents the optimal MV portfolios used to calculate the Shapley value of the assets. Since the MV efficient frontier is a function of the required mean return μ_p, the variance of a frontier portfolio is provided by Eq. (7), which can be written equivalently as

$$\sigma_p^2 = \frac{1}{D}(C\mu_p^2 - 2A\mu_p + B). \tag{8}$$

Then, for an arbitrary set of required mean returns μ_p, using Eq. (8) we calculate the frontier portfolio variance for each subset $S \cup i \subseteq N$. The Shapley value is computed following Eq. (5) using the variance-covariance matrix Σ_S and the quadratic forms $A_S = \mathbf{1}_S{}' \Sigma_S{}^{-1} \mu_S$, $B_S = \mu_S{}' \Sigma_S{}^{-1} \mu_S$, $C_S = \mathbf{1}_S{}' \Sigma_S{}^{-1} \mathbf{1}_S$, and $D_S = B_S C_S - A_S^2$ for all the 2^N subsets $S \subseteq N$. The Shapley value for each stock i of an optimal frontier portfolio subject to a given mean μ_p is obtained as

$$Sh_i(\sigma_p^2; \mu_p) = \sum_{s=0}^{N-1} \sum_{S \subset N \backslash i} \frac{(n-s-1)!\, s!}{n!} [\sigma_p^2(\mu_p, S \cup \{i\}) - \sigma_p^2(\mu_p, S)] \quad \forall i \in N. \quad (9)$$

Weighted Shapley values for each asset on a optimal frontier portfolio are computed by adding the weight ratio given by Eq. (4) to formula (9) as follows:

$$Shw_i(\sigma_p^2; \mu_p) = \sum_{s=0}^{N-1} \sum_{S \subset N \backslash i} \frac{(n-s-1)!\, s!}{n!} \varphi_i [\sigma_p^2(\mu_p, S \cup \{i\}) - \sigma_p^2(\mu_p, S)] \quad \forall i \in N. \quad (10)$$

Finally, for a given return μ_p, the weighted Shapley values sum up to their optimal portfolio variance at μ_p as

$$\sum_{i=1}^{N} Shw_i(\sigma_p^2; \mu_p) = \sigma_p^2(\mu_p). \quad (11)$$

It seems natural to now discuss the Shapley value as expressed by Eq. (9) for an asset in an optimal portfolio. Given that efficient portfolios have the lowest variance for a given mean, the incremental risks $\sigma_p^2(\mu_p, S \cup \{i\}) - \sigma_p^2(\mu_p, S)$ are non-positive for any asset i and any set S that does not contain i . Indeed as assets are added to the portfolio the variance does not increase. However, the Shapley value computation also includes the incremental risk of going from an empty portfolio to a portfolio of one asset i whose increment is usually positive. Hence, as it is shown in the empirical analysis, Shapley values of assets in optimal portfolios can be either negative or positive. Negative Shapley values imply that these assets reduce their risk contribution to the portfolio as mean return increase. Positive Shapley values imply increasing risk assets along the efficient frontier and therefore increase mean return.

4. The empirical application

I now present the empirical analysis of computing the weighted Shapley values of assets in MV efficient portfolios. For that purpose I have collected the daily returns of the 13 most traded stocks from the Dow-Jones Industrial Average during the year 2020[3]. In addition to daily returns I have collected the daily trading volume and computed its mean. The summary statistics of the collected data are presented in Table 1.

My contention is that asset size affects risk valuation. Hence, to characterize the importance of an asset in a portfolio, the relative asset trading volume is introduced in the risk valuation as implied by the relative size of assets expressed by Eq. (4). Since the onset of CAPM, it was theoretically established that the entire universe of risky assets, i.e., represented by the market portfolio, was the main sole determinant for the systematic risk of individual securities. Today, analysts can sensibly assert that additional factors affect systematic risk because otherwise the basic relationship of the market model equation[4] can be tested only with great difficulty. To improve this relationship many financial analysts have added explanatory variables to the basic equation. Oddly enough, it appears from the finance literature that trading volume can either affect systematic risk [4] or, alternatively, systematic risk can affect trading volume [7].

The efficient frontier for these stocks is constructed as follows: First, the means μ and the variance-covariance matrix Σ are computed. Then, the quadratic forms A, B, C, and D are obtained and the MV efficient frontier is calculated using Eq. (7) for six arbitrary means. The optimal weights of the assets are reported in Table 2 for the six portfolios and the efficient frontier is depicted in Fig. 1 for the space mean-standard deviation.

Let us now analyze the portfolio components as we move on the efficient frontier from a low risk-low mean return portfolio such as portfolio II to a high risk-high mean return portfolio such as VI. While the shares of some long held assets such as AAPL and DIS increased along the efficient frontier, the short held assets such as BA and INTC have their positions further decreased. On the other hand, JNJ and WMT have substantial long positions that hardly change when moving along the efficient frontier. Although there is only a small set of assets on the efficient frontier we still are able to attain a diversified universe that will provide an interesting display of weighted Shapley values as follows.

[3] Because of the dimensionality of the 2^N subsets and the limitations of any known computer algorithm, I cannot, for the present, evaluate the Shapley values when N exceeds 13.

[4] $r_k = \alpha_k + \beta_k \, r_M$ where r_k *and* r_M are the asset and the market returns.

Table 1
13 stocks DJIA daily returns statistics 2020

Symbol	Mean	Std dev	Mean value of trading volume	Weights
AAPL	0.28%	2.94%	$13,689,867,976	0.3412
BA	−0.01%	5.51%	$5,028,359,616	0.1253
CRM	0.18%	3.33%	$1,536,718,998	0.0383
CSCO	0.02%	2.64%	$1,036,497,778	0.0258
DIS	0.14%	3.09%	$1,790,549,678	0.0446
HD	0.13%	2.74%	$1,115,412,898	0.0278
INTC	−0.01%	3.36%	$1,669,569,574	0.0416
JNJ	0.06%	1.91%	$1,140,413,071	0.0284
JPM	0.04%	3.42%	$1,932,329,472	0.0482
MSFT	0.18%	2.76%	$7,021,122,760	0.1749
UNH	0.12%	3.03%	$1,165,555,850	0.0290
V	0.10%	2.69%	$1,836,569,770	0.0458
WMT	0.10%	1.98%	$1,160,240,486	0.0289

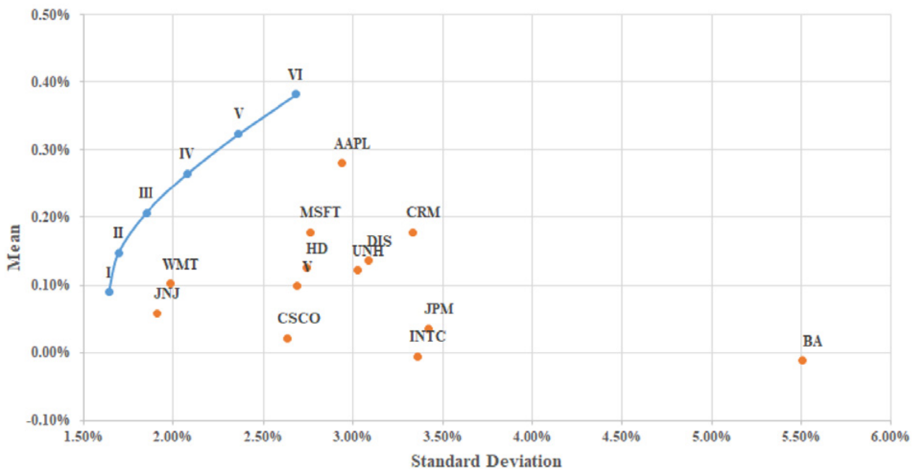

Fig. 1. Mean-variance efficient frontier for 13 stocks.

In Table 3 I present the weighted Shapley values for the assets composing the portfolios on the efficient frontier according to Eq. (6). The values are expressed in terms of the standard deviation of the optimal portfolios. From the table, we observe that some Shapley values are positive while others are negative, indicating that this specific asset when reduces substantially portfolio risk. As we are dealing with efficient frontier portfolios it implies that this reducing risk also reduces the expected return. For example, if we concentrate on portfolio IV we remark that heavier traded stocks such as AAPL and MSFT have a negative weighted Shapley value reducing risk whereas some lighted traded securities such a HD and UNH have a larger positive weighted Shapley value that contribute more to the portfolio risk. We now compare these results with the standard Shapley values exhibited on Table 4.

Standard Shapley values of assets on the efficient frontier are computed using Eq. (9) for the same portfolios defined in Table 2. Some standard Shapley values are negative like AAPL as with weighted values but MSFT exhibits positive Shapley values. We remark that the Shapley values of some large stocks (AAPL) and some small ones (CSCO) unexpectedly decline when moving along the efficient frontier from lower risk to higher risk portfolios. On the other hand as expected, the Shapley values of large stocks such as MSFT increase along the frontier. When we compare

Table 2
Assets Weights of efficient frontier portfolios

Portfolio	Mean ↓	Std Dev ↓	I	II	III	IV	V	VI
Mean →			0.09%	0.15%	0.21%	0.27%	0.32%	0.38%
Std Dev →			1.64%	1.69%	1.85%	2.08%	2.36%	2.68%
AAPL	0.28%	2.94%	1.29%	18.26%	35.23%	52.20%	69.17%	86.14%
BA	−0.01%	5.51%	-2.97%	−5.27%	−7.58%	−9.89%	−12.19%	−14.50%
CRM	0.18%	3.33%	9.32%	8.76%	8.19%	7.63%	7.06%	6.50%
CSCO	0.02%	2.64%	−6.94%	−17.32%	−27.70%	−38.09%	−48.47%	−58.85%
DIS	0.14%	3.09%	15.06%	22.17%	29.29%	36.41%	43.52%	50.64%
HD	0.13%	2.74%	9.51%	11.11%	12.70%	14.29%	15.89%	17.48%
INTC	−0.01%	3.36%	−5.48%	−12.59%	−19.71%	−26.83%	−33.94%	−41.06%
JNJ	0.06%	1.91%	61.15%	55.24%	49.34%	43.43%	37.52%	31.62%
JPM	0.04%	3.42%	−3.46%	−5.43%	−7.39%	−9.36%	−11.32%	−13.28
MSFT	0.18%	2.76%	−24.02%	−20.44%	−16.87%	−13.30%	−9.73%	−6.15%
UNH	0.12%	3.03%	−7.40%	−5.19%	−2.98%	−0.77%	1.44%	3.65%
V	0.10%	2.69%	6.41%	3.10%	−0.21%	−3.52%	−6.83%	−10.14%
WMT	0.10%	1.98%	47.52%	47.61%	47.70%	47.79%	47.88%	47.97%

Table 3
Weighted Shapley values of optimal portfolios assets

Portfolio	Mean	Std Dev	I	II	III	IV	V	VI	Weights
Mean →			0.09%	0.15%	0.21%	0.27%	0.32%	0.38%	
Std Dev			1.64%	1.69%	1.85%	2.08%	2.36%	2.68%	
AAPL	0.28%	2.94%	−0.01%	−0.05%	−0.18%	−0.35%	−0.54%	−0.73%	0.3412
BA	−0.01%	5.51%	0.01%	0.00%	−0.07%	−0.14%	−0.21%	−0.28%	0.1253
CRM	0.18%	3.33%	0.08%	−0.16%	−0.65%	−1.05%	−1.44%	−1.82%	0.0383
CSCO	0.02%	2.64%	0.20%	0.35%	−0.18%	−0.74%	−1.29%	−1.84%	0.0258
DIS	0.14%	3.09%	0.15%	−0.11%	−0.12%	−0.10%	−0.07%	−0.03%	0.0446
HD	0.13%	2.74%	0.76%	0.45%	1.33%	2.23%	3.14%	4.06%	0.0278
INTC	−0.01%	3.36%	0.08%	0.07%	−0.39%	−0.85%	−1.31%	−1.76%	0.0416
JNJ	0.06%	1.91%	−0.35%	−0.03%	−0.31%	−0.65%	−0.99%	−1.35%	0.0284
JPM	0.04%	3.42%	0.12%	0.30%	0.13%	−0.04%	−0.20%	−0.35%	0.0482
MSFT	0.18%	2.76%	0.06%	−0.01%	−0.11%	−0.17%	−0.24%	−0.32%	0.1749
UNH	0.12%	3.03%	0.83%	0.61%	1.61%	2.62%	3.65%	4.68%	0.0290
V	0.10%	2.69%	−0.01%	0.27%	0.49%	0.71%	0.95%	1.18%	0.0458
WMT	0.10%	1.98%	−0.29%	0.01%	0.30%	0.61%	0.91%	1.22%	0.0289

the weighted Shapley values with the standard ones we observe that the weighted Shapley values are in general more moderate values and do not exhibit extreme values, implying that using the relative size of an asset in the market seems to improve risk valuation. The correction brought about by weighted Shapley values is worthwhile especially when one observes the stocks with large trading volume such as AAPL and MSFT. Indeed, for optimal portfolio VI for example, its standard deviation 2.68%. AAPL's standard Shapley value is negative 4.58% where MSFT's Shapley value is positive 9.60%. The values seems extreme. Now with the weighted Shapley value the figures are more sensible with AAPL as negative 0.73% and MFST negative 0.32%.

5. Concluding remarks

In this paper I have decomposed and computed the risk of optimal portfolios attributing to each asset their fair share using the concept of weighted Shapley values. Calculating Shapley values for large portfolios is challenging as it becomes exponentially cumbersome. The notion of weighted Shapley value is rewarding because

Table 4
Shapley values of optimal portfolios assets

Portfolio	Mean	Std Dev	I	II	III	IV	V	VI
Mean →			0.09%	0.15%	0.21%	0.27%	0.32%	0.38%
Std Dev →			1.64%	1.69%	1.85%	2.08%	2.36%	2.68%
AAPL	0.28%	2.94%	−0.85%	−0.53%	−1.07%	−2.25%	−3.42%	−4.58%
BA	−0.01%	5.51%	−0.01%	0.89%	1.12%	0.77%	0.43%	0.10%
CRM	0.18%	3.33%	4.91%	1.51%	1.17%	4.18%	7.21%	10.24%
CSCO	0.02%	2.64%	−0.79%	−0.07%	−0.13%	−0.82%	−1.53%	-2.24%
DIS	0.14%	3.09%	−0.72%	−0.41%	−0.40%	−0.93%	−1.45%	−1.96%
HD	0.13%	2.74%	−0.67%	−0.31%	−0.01%	−0.29%	−0.56%	−0.82%
INTC	−0.01%	3.36%	−0.26%	0.63%	0.74%	0.23%	−0.28%	−0.79%
JNJ	0.06%	1.91%	−1.31%	−0.53%	−0.49%	−1.10%	−1.74%	-2.39%
JPM	0.04%	3.42%	−0.70%	−0.01%	−0.02%	−0.63%	−1.24%	−1.84%
MSFT	0.18%	2.76%	4.83%	1.42%	0.96%	3.82%	6.71%	9.60%
UNH	0.12%	3.03%	−0.62%	−0.22%	0.11%	−0.13%	−0.36%	−0.59%
V	0.10%	2.69%	−0.88%	−0.18%	0.02%	−0.37%	−0.76%	−1.14%
WMT	0.10%	1.98%	−1.28%	−0.50%	−0.14%	−0.39%	−0.64%	−0.90%

it removes the need for symmetry assumption of assets regulating the various coalitions. Financial markets are diverse enough to have larger tradeable securities and smaller less liquid assets. Standard Shapley valuation that forego this feature may bias the mean adjusted risk attribution of securities on the efficient frontier portfolios. Weighted Shapley values may remedy this lacuna. The question of course remains as what is the best statistic that defines weights in the Shapley value computation. In the present paper I have used trading volume, which is a variable that has shown in the past to considerably improve the computation of systematic risk.

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